

# On Hegerfeldt's paradox

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The acausal behavior of relativistic states exhibited by Hegerfeldt is shown not to be present in physical systems described by first order in time evolution equations.

The question of whether relativistic quantum theory is causal or not cannot be considered settled, as a recent controversy between Hegerfeldt [1] and Buchholz and Yngvason [2] shows. In extension of earlier results [3,4] Hegerfeldt [1] considered Fermi's [5] two-atom system, where an atom  $A$  is in an excited state, and an atom  $B$ , which is separated from  $A$  by some distance  $R$ , is in its ground state and one looks at which time an excitation of  $B$  may occur. Hegerfeldt gave mathematical proof, that even if the state of  $A$  is localized at  $t = 0$ , an excitation of  $B$  occurs with finite probability at any time  $t > 0$ , not only for times  $t > R/c$ , as one expects from Einstein causality, i.e. that signal propagation is limited by the velocity of light  $c$ . This violation of causality, originally proven for solutions of the Klein-Gordon equation [3], we call "Hegerfeldt's paradox". In their reply Buchholz and Yngvason [2] argued on the general grounds of algebraic quantum field theory (AQFT) that there no paradox with causality exists, and questioned the assumptions on localization Hegerfeldt employed. Hegerfeldt takes the standpoint that his framework goes beyond that of Buchholz and Yngvason, and that the restrictions inherent to the framework of AQFT render it inapplicable to the Fermi two atom system [6]. Hence the question raised is whether quantum field theory in its axiomatic [7] and algebraic [8] foundations, where causality is well-established, is a sufficient basis for applications in interacting physical systems. In this short article we want to comment on the restrictions inherent to both lines of argumentation.

We first consider classical relativistic fields. The initial value problem for the Klein-Gordon equation

$$\left[ \frac{\partial^2}{\partial t^2} - \sum_{i=1}^3 \frac{\partial^2}{\partial x^{i2}} + m^2 \right] \Phi(t, \mathbf{x}) = 0, \quad (1)$$

with Cauchy data given at  $t = t_0$ , is solved by (notation:  $x = (t, \mathbf{x})$ ,  $\mathbf{x} = (x^1, x^2, x^3)$ ,  $c = 1$ )

$$\Phi(x') = \int \frac{d^4x}{\sqrt{8\pi}} \Delta(x - x'; m^2) (\delta'(t - t_0) \Phi(t_0, x) + \delta(t - t_0) \partial_t \Phi(t_0, \mathbf{x})) \quad (2)$$

where  $\delta$  is the Dirac delta distribution and

$$\Delta(x; m^2) = \Delta_+(x; m^2) - \Delta_+(-x; m^2) \quad (3)$$

with

$$\Delta_+(x; m^2) = \frac{i}{2(2\pi)^3} \int e^{-itp^0 + i\mathbf{x}\cdot\mathbf{p}} \frac{d^3p}{p^0}, \quad p^0 = \sqrt{\mathbf{p}^2 + m^2} \quad (4)$$

is the causal propagator for the Klein-Gordon equation.  $\Delta$  is it a tempered distribution [9] (Th. IX.47), with support contained in the forward and backward lightcones [9] (Th. IX.48).

We see from (2) that the solutions of the Klein-Gordon equation behave causally with respect to the joint support of  $\Phi(t_0, \mathbf{x})$  and  $\partial_t \Phi(t_0, \mathbf{x})$ . This is an induction phenomenon well-understood in Maxwell theory. In absence of matter, all components of the electric and magnetic fields satisfy the wave equation (1) for  $m = 0$ . The time derivatives of the electromagnetic field components are interrelated through the Maxwell equations, and (2) exhibits the induction of an electric field by a magnetic field, and vice versa. Similar, the first order Dirac equation provides a relation between the spinor components and their time derivatives, while all spinor components satisfy the Klein-Gordon equation (1). In general, if a multicomponent state  $\Psi$  is governed by a Hamiltonian  $H$ ,

$$i\partial_t \Psi = H\Psi$$

and the Hamiltonian is local in the sense that

$$\text{supp}(H\Psi) \subset \text{supp} \Psi$$

then the time evolution is causal with respect to the joint support of all components of  $\Psi$ ; in this case Hegerfeldt's considerations do not apply.

The basis of Hegerfeldt's arguments is the nonlocal nature of the Hamiltonian  $\sqrt{\mathbf{p}^2 + m^2}$  for scalar fields when spectral positivity is assumed. Then we cannot have compact supports both of  $\Phi(t_0, \mathbf{x})$  and  $\partial_t \Phi(t_0, \mathbf{x})$ . If  $\Phi(t_0, \mathbf{x})$  has compact support, then it has an analytic three-dimensional Fourier transform,  $\hat{\Phi}(\mathbf{p})$ , but since  $\sqrt{p^2 + m^2} \hat{\Phi}(\mathbf{p})$ , the Fourier of  $\partial_t \Phi(t_0, \mathbf{x})$  with spectral positivity assumed, is not analytic,  $\partial_t \Phi(t_0, \mathbf{x})$  cannot have compact support. Looking at the support of  $\Phi(t_0, \mathbf{x})$  alone, positive frequency solutions appear to behave acausal [3]. One implication is, that positive frequency solutions of first order relativistic wave equation equations never have compact support [10]: Hegerfeldt's results cannot be applied to a system of coupled Fermions and photons as is Fermi's two atom system. Moreover we see a problem to interpret the scalar case consistently. If  $\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*$  is interpreted as charge density, with the notion that a particle can be located by measurement only where this quantity is nonzero, we get into conflict with interpreting  $|\partial_t \Phi|^2$  as energy density since this latter quantity is not located where the charge density is; and we cannot speak of energy inducing charge. In the case of uncharged particles the quantity  $\Phi^* \partial_t \Phi - \Phi \partial_t \Phi^*$  cannot be interpreted as probability density, since a particle can be located only through its response to external forces. Since the problems pertain to interacting theories [4] we conclude that a classical theory to be consistent and causal must not have charged scalar fields. It is interesting to note that in the framework of general quantum fields theory [8] a corresponding problem related to causality and the charge structure of a free field theory is well known. As Haag points out [8] (end of Sec. III), the current cannot be defined as local observable even in free Dirac theory.

In a free quantum field theory the solutions of the classical field equations form the single-particle Hilbert space. Hence the support properties are identical and the Hamiltonian of a scalar field is not a local operator in the sense used above. In the axiomatic framework, the notion of locality is always related to the support of the test functions, but this support is not related the support of the Hilbert space states in configuration representation. Test functions are Fourier transformed and projected onto mass shells, whereby the compact support in general is lost for the inverse Fourier-transformed state.

Where Hegerfeldt concludes that the theory has a problem with causality, we see already from the classical theory that the support properties of Fermion and photon states will not allow for the existence of space-like separated, localized states needed for the conclusion. Buchholz and Yngvason [2] pointed out that no such states with finite energy exist in algebraic quantum field theory. The theory is fully causal, but may not contain the desired local observables or localized states, which is precisely Hegerfeldt's point of criticism [6].

With respect to Fermi's two atom system we conclude, that the assumption of spectral positivity leads to the result, that in any experimental preparation the state of an excited atom A will always have a finite overlap to the state of the non-excited atom B, which allows for causal excitation without time delay. For experimental purposes, this overlap is relevant only for distances comparable to the Compton wave lengths of the constituents, since these are the characteristic lengths of the exponential tails of the states [11].

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